SEMI-PRIMARY HEREDITARY RINGS

BY

A. ZAKS

ABSTRACT

A semi-primary hereditary ring Σ, with radical M and residue ring $\Gamma = \Sigma/M$, is uniquely determined by Γ and a Γ-bimodule $A = M/M^2$, whenever Σ admits a splitting $\Sigma = \Gamma + A + M^2$.

The purpose of this paper is twofold. We first generalize the main theorem of Jans and Nakayama [3] concerning a semi primary ring that admits a splitting, $\Lambda = \Gamma + A + N^2$, and such that gl.dim $\Lambda/N^2 < \infty$. It turns out that the splitting assumption is superfluous. Next we discuss the extent to which Ω is unique, where Ω is a semi primary hereditary ring of which Λ is a residue.

We say that Λ is a *semi primary ring*, if its (Jacobson) radical N is nilpotent, and the residue ring Γ, $\Gamma = \Lambda/N$, is a semi simple (Artinian) ring.

If $e_o, \cdots, e_k$ are $k+1$ idempotents in Λ, then we say that the sequence $(e_o, \cdots, e_k)$ is a *connected sequence of length* k if $e_i N e_{i+1} \neq 0$ for all i, $i = o, \cdots, (k-1)$.

We denote by $l(\Lambda)$ the supremum of the length of connected sequences, in Λ. If I is any two-sided ideal in Λ such that $I \subset N^2$ then $l(\Lambda/I) = l(\Lambda)$. Jans and Nakayama have shown in [3] that gl.dim$\Lambda/N^2 = l(\Lambda)$. If, furthermore, Λ admits a splitting, $\Lambda = \Gamma + A + N^2$, then gl.dim$\Lambda/I \leqq$ gl.dimΛ/N^2, and Λ is a residue ring of a semi-primary hereditary ring Ω (i.e. gl.dim$\Omega \leqq 1$).

If $l(\Lambda) < \infty$, Chase has shown in [1] that Λ is triangular, i.e. one can arrange every complete set $(e_1, \cdots, e_t)$ of mutually orthogonal idempotents in Λ, $\Lambda = \Lambda e_1 \oplus \cdots \oplus \Lambda e_t$, so that $e_i N e_j = 0$ whenever $i < j$.

Denote by $n(\Lambda)$ the integer satisfying $N^{n(\Lambda)} = 0$ and $N^{n(\Lambda)-1} \neq 0$. Then $n(\Lambda) - 1 \leqq l(\Lambda)$ whenever gl.dim$\Lambda/N^2 < \infty$ (e.g. [1], [3]).

If A is any (two-sided) Γ-bimodule, then one constructs the ring $\Omega(\Gamma, A) = \Gamma + A + A \otimes_\Gamma A + A \otimes_\Gamma A \otimes_\Gamma A + \cdots$, this turned out to be very useful in studying semi-primary rings all of whose residue rings have finite global dimension.

Unless otherwise specified let Λ denote a semi primary ring with radical N, such that gl.dim$\Lambda/N^2 < \infty$. Hence Λ is triangular [1]. An immediate consequence is the existence of a simple projective module. Furthermore, since $e_1 N e_j = 0$ for

Received 11 July, 1968.

$j = 1, \cdots, t$, it follows by straightforward computation that $\Lambda e_1 / N e_1$ is a simple injective module.

THEOREM 1. *gl.dim*$\Lambda/I \leqq$ *gl.dim*Λ/N^2 *for every two sided ideal, I, such that* $I \subset N^2$.

Proof. Λ admits a splitting, $\Lambda = \Gamma + N$, and Λ is a residue of the semi primary hereditary ring $\Omega = \Omega(\Gamma, N)$. If K is an ideal in Ω we denote by $K^{(t)}$ the tth power of K in Ω. We have $N^{(l(\Lambda)+1)} = 0$. Let J be the inverse image of I in Ω under the canonical projection of Ω-onto Λ, then Ω/J is isomorphic to Λ/I.

It now follows from [3, Prop. 9] that gl.dim$\Lambda/I \leqq l(\Lambda)$. Since gl.dim$\Lambda/N^2 = l(\Lambda)$ we finally obtain the inequality gl.dim$\Lambda/I \leqq$ gl.dimΛ/N^2.

For the rest let Λ be a semi primary ring with radical N of square zero, such that gl.dim$\Lambda = n$. Then if $\Omega = \Omega(\Gamma, N)$ it follows that $N^{(n)} \neq 0$ and $N^{(n+1)} = 0$.

If Σ is any semi primary hereditary ring with radical M, and Σ/M^2 is isomorphic to Λ then $n = l(\Lambda) = l(\Sigma)$, $M^n \neq 0$, and $M^{n+1} = 0$. This follows from the following lemma:

LEMMA 2. *If Σ is a semi primary hereditary ring, then $n(\Sigma) = l(\Sigma) + 1$.*

COROLLARY 3. *Let $\Sigma_1(\Sigma_2)$ be a semi-primary hereditary ring with radical $N_1(N_2)$. If Σ_1/N_1^2 is isomorphic to Σ_2/N_2^2 then $l(\Sigma_1) = l(\Sigma_2)$ and $n(\Sigma_1) = n(\Sigma_2)$.*

If 0 and 1 are the unique central idempotents in $\Sigma_1(\Sigma_2)$, then the center of $\Sigma_1(\Sigma_2)$ is a field, say $F_1(F_2)$ (e.g., [4]). The center of Σ_1/N_1^2 is a field, say F, and up to an isomorphism F_1 and F_2 are subfields of F. Furthermore, the semi primary hereditary ring $\Omega(\Sigma_1/N_1, N_1)$ has F as its center. The following example shows that it is possible that F_1 is a proper subfield of F.

EXAMPLE. Let x be a transcendental element over a field k of characteristic 2. Set $R = k(x^{1/2}) \otimes_{k(x)} k(x^{1/2})$. Let Σ, be the subring of the 3×3 matrix algebra over the ring R. Denote by N the radical of R. Denote by $K_1(K_2)$ the canonical embedding of $k(x^{1/2})$ in the first (second) factor of R. A matrix σ belongs to Σ_1 iff σ has the form

$$\begin{Vmatrix} a & o & o \\ b & c & o \\ d & e & f \end{Vmatrix}$$

where $a \in K_1$, $e, c, f \in K_2$, $b \in N$, and $d \in R$.

The center of Σ_1 can be easily checked to be isomorphic to $k(x)$. Furthermore, Σ_1 is an Artinian hereditary ring. Finally, by straightforward computations one verifies that the center of Σ_1/N_1^2 is isomorphic to $k(x^{1/2})$. Thus the center of $\Omega(\Sigma_1/N_1, N_1)$ is $k(x^{1/2})$.

If R is a semi primary ring with radical V such that gl.dim $R/V^2 < \infty$, then R is a residue of a semi primary hereditary ring Σ. We don't know if there exists a ring Σ for which Σ/M^2 is isomorphic to R/V^2, but there exists a ring Σ for which the following equalities hold: $l(\Sigma) = l(R)$, $n(\Sigma) = n(R)$, and the center of Σ is isomorphic to the center of R. Furthermore, Σ admits a splitting, $\Sigma = \Gamma + A + M^2$.

If R is a semi primary ring with radical V that admits a splitting $R = \Gamma + A + V^2$, where Γ is isomorphic to R/V, and if gl.dim $R/V^2 < \infty$ then R is a residue ring of the semi primary hereditary ring $\Omega(\Gamma, A)$ [3]. Let Σ be a semi primary hereditary ring such that R is a residue ring of Σ, and Σ/M^2 is isomorphic to R/V^2 where M is the radical of Σ. Σ need not admit a splitting [4], but:

THEOREM 4. *If Σ admits a splitting $\Sigma = \Gamma + A + M^2$, then Σ is isomorphic to $\Omega(\Gamma, A)$.*

Proof. Since $\Omega(\Gamma, A) = \Gamma + A + A \otimes_\Gamma A + \cdots$, the map obtained by the obvious extenstion of the identity on $\Gamma + A$ to a map from $\Omega(\Gamma, A)$ to Σ is a well defined ring homomorphism, say ϕ. Set $M_0 = A + A \otimes_\Gamma A + \cdots$, then $\phi(M_0)$ is an ideal in Σ that contains A and is contained in M. Therefore $M = \phi(M_0) + M^2$, and this readily implies that $\phi(M_0) = M$. Thus ϕ is an epimorphism. Since ϕ is an isomorphism on $\Gamma + A$, it follows that $\ker \phi$ is contained in M_0^2. But gl.dim $\Omega/\ker \phi = 1$ and $\ker \phi \, c \, M_0^2$ imply that $\ker \phi = 0$, thus ϕ is an isomorphism.

If the center of Σ is the field F, then a splitting of Σ, $\Sigma = \Gamma + A + M^2$, will result if $\dim_F \Gamma = 0$. We thus have:

COROLLARY 5. *If Σ is an hereditary semi primary ring with center F, and if $\dim_F \Gamma = 0$, then Σ is isomorphic to $\Omega(\Gamma, A)$ where A is isomorphic to M/M^2.*

Remark that if for every semi primary hereditary ring Ω with radical M_1, for which Ω/M_1 is isomorphic to Γ, and for which F is in the center of Ω, it follows that Ω admits a splitting, $\Omega = \Gamma + A_1 + M_1^2$, then necessarily $\dim_F \Gamma = 0$ (e.g., [4]).

It turns out that every semi primary hereditary ring Λ is a residue of a semi primary hereditary ring Ω that admits a splitting, $\Omega = \Gamma + A + M^2$. It might be of some interest to describe the ideals I in Ω for which gl.dim $\Omega/I = 1$.

PROPOSITION 6. *Let R be a ring satisfying the following conditions:*

(a) *R contains a two-sided ideal I, such that $\cap_m I^m = (o)$;*

(b) *R/I is a semi-simple (Artinian) ring.*

(c) *$l \cdot$ gl.dim $R/I^2 < \infty$.*

then R is a semi primary ring with radical I.

Proof. Since R/I^{k+1} is a semi-primary ring with radical I/I^{k+1} such that $n(I/I^{k+1}) = k + 1$, then $l(R/I^{k+1}) \geqq k$. Observing that $l(R/I^{k+1}) = l(R/I^2) < \infty$ the result follows immediately.

REFERENCES

1. S. U. Chase, *A generalization of the ring of triangular matrices*, Nagoya Math. J., **18** (1961), 13–25.

2. S. Eilenberg, H. Nagao and T. Nakayama, *Dimension of Residue Rings of Hereditary Rings*, Nagoya Math. Jour., **10** (1956), 87–95.

3. J. P. Jans and T. Nakayama, *Algebras with Finite Dimensional Residue Algebras*, Nagoya Math. Jour., **11** (1957), 67–76.

4. A. Zaks, *A note on semi primary hereditary rings*, Pac. J. Math., **23** (1967), 627–628.

TECHNION–ISRAEL INSTITUTE OF TECHNOLOGY,
HAIFA